Equitable colorings of complete multipartite graphs

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Abstract

A q-*equitable coloring* of a graph G is a proper q-coloring such that the sizes of any two color classes differ by at most one. In contrast with ordinary coloring, a graph may have an equitable q-coloring but has no equitable $(q+1)$-coloring. The *equitable chromatic threshold* is the minimum p such that G has an equitable q-coloring for every $q \geq p$.

In this paper, we establish the notion of $p(q : n_1, \ldots, n_k)$ which can be computed in linear-time and prove the following. Assume that $K_{n_1,\ldots,n_k}$ has an equitable q-coloring. Then $p(q : n_1, \ldots, n_k)$ is the minimum p such that $K_{n_1,\ldots,n_k}$ has an equitable r-coloring for each r satisfying $p \leq r \leq q$. Since $K_{n_1,\ldots,n_k}$ has an equitable $(n_1+\cdots+n_k)$-coloring, the equitable chromatic threshold of $K_{n_1,\ldots,n_k}$ is $p(n_1 + \cdots + n_k : n_1, \ldots, n_k)$.

We find out later that the aforementioned immediate consequence is exactly the same as the formula of Yan and Wang [12]. Nonetheless, the notion of $p(q : n_1, \ldots, n_k)$ can be used for each q in which $K_{n_1,\ldots,n_k}$ has an equitable q-coloring and the proof presented here is much shorter.

1 Introduction

Throughout this paper, all graphs are finite, undirected, and simple. We use $V(G)$ and $E(G)$, respectively, to denote the vertex set and edge set of a graph G. Let $K_{n_1,\ldots,n_k}$ be a complete k-partite graph in which partite set X_i has size n_i for $1 \leq i \leq k$. Let K_{k*n} denote a complete k-partite set with each partite set has size n.

An *equitable k-coloring* of a graph is a proper vertex k-coloring such that the sizes of every two color classes differ by at most 1.

It is known [3] that determining if a planar graph with maximum degree 4 is 3-colorable is NP-complete. For a given n-vertex planar graph G with maximum degree 4, let G' be the graph obtained from G by adding $2n$ isolated vertices. Then G has 3-coloring if and only if

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G' has an equitable 3-coloring. Thus, finding the minimum number of colors needed to color a graph equitably even for a planar graph is an NP-complete problem.

Hajnal and Szemerédi [4] settled a conjecture of Erdős by proving that every graph G with maximum degree at most Δ has an equitable k-coloring for every $k \geq 1 + \Delta$. This result is now known as Hajnal and Szemerédi Theorem. Later, Kierstead and Kostochka [5] gave a simpler proof of Hajnal and Szemerédi Theorem. The bound of the Hajnal-Szemerédi theorem is sharp, but it can be improved for some important classes of graphs. In fact, Chen, Lih, and Wu [1] put forth the following conjecture.

Conjecture 1 *Every connected graph G with maximum degree $\Delta \geq 2$ has an equitable coloring with Δ colors, except when G is a complete graph or an odd cycle or Δ is odd and $G = K_{\Delta,\Delta}$.*

Lih and Wu [8] proved the conjecture for bipartite graphs. Meyer [9] proved that every forest with maximum degree Δ has an equitable k-coloring for each $k \geq 1 + \lceil \Delta/2 \rceil$ colors. This result implies the conjecture holds for forests. Yap and Zhang [13] proved that the conjecture holds for outerplanar graphs. Later Kostochka [6] improved the result by proving that every outerplanar graph with maximum degree Δ has an equitable k-coloring for each $k \geq 1 + \lceil \Delta/2 \rceil$.

In [15], Zhang and Yap essentially proved the conjecture holds for planar graphs with maximum degree at least 13. Later Nakprasit [10] extended the result to all planar graphs with maximum degree at least 9. Some related results are about planar graphs without some restricted cycles [7, 11, 16].

Moreover, the conjecture has been confirmed for other classes of graphs, such as graphs with degree at most 3 [1, 2] and series-parallel graphs [14].

In contrast with ordinary coloring, a graph may have an equitable k-coloring but has no equitable $(k+1)$-coloring. For example, $K_{7,7}$ has an equitable k-coloring for $k = 2, 4, 6$ and $k \geq 8$, but has no equitable k-coloring for $k = 3, 5$ and 7. This leads to the definition of the *equitable chromatic threshold* which is the minimum p such that G has an equitable q-coloring for every $q \geq p$.

In this paper, we establish the notion of $p(q : n_1, \ldots, n_k)$ which can be computed in linear-time and prove the following. Assume that $K_{n_1,\ldots,n_k}$ has an equitable q-coloring. Then $p(q : n_1, \ldots, n_k)$ is the minimum p such that $K_{n_1,\ldots,n_k}$ has an equitable r-coloring for each r satisfying $p \leq r \leq q$. Since $K_{n_1,\ldots,n_k}$ has an equitable $(n_1 + \cdots + n_k)$-coloring, the equitable chromatic threshold of $K_{n_1,\ldots,n_k}$ is $p(n_1 + \cdots + n_k : n_1, \ldots, n_k)$.

We find out later that the aforementioned immediate consequence is exactly the same as the formula of Yan and Wang [12]. Nonetheless, the notion of $p(q : n_1, \ldots, n_k)$ can be used for each q in which $K_{n_1,\ldots,n_k}$ has an equitable q-coloring and the proof presented here is much shorter.

2 Main Result

We introduce the notion of $p(q : n_1, \ldots, n_k)$ which can be computed in linear-time.

Definition 1 *Assume that $K_{n_1,\dots,n_k}$ has an equitable q-coloring, and d is the minimum value not less than $\lceil (n_1+\cdots+n_k)/q \rceil$ such that (i) there are distinct i and j in which n_i and n_j are not divisible by d, or (ii) there is n_j with $n_j/\lfloor n_j/d \rfloor > d+1$. Define $p(q : n_1,\dots,n_k) = \lceil n_1/d \rceil + \cdots + \lceil n_k/d \rceil$.*

Lemma 1 *Assume that $G = K_{n_1,\dots,n_k}$ has an equitable q-coloring. Then G has an equitable r-coloring for each r satisfying $p(q : n_1,\dots,n_k) \le r \le q$.*

Proof. Let $p = p(q : n_1,\dots,n_k)$ and $N = n_1+\cdots+n_k$. We prove by reverse induction that G has an equitable r-coloring for each r satisfying $p \le r \le q$. By assumption, G has an equitable q-coloring. Consider r in which $p < r \le q$ and G has an equitable r-coloring f. We show that G has an equitable $(r-1)$-coloring. Let $b = \lceil N/r \rceil$. By assumption, there are integers r_i and s_i such that f partitions X_i into $r_i - s_i$ color classes of size b and s_i color classes of size $b-1$ where $r = r_1+\cdots+r_k$. Thus $n_i = (r_i - s_i)b + s_i(b-1) = r_i b - s_i$ for each i.

CASE 1: Some j has $r_j \ne \lceil n_j/b \rceil$. Note that $n_j = \lceil n_j/b \rceil b - g_j$ for some g_j satisfying $0 \le g_j \le b-1$. Now, we have $r_j b - s_j = \lceil n_j/b \rceil b - g_j$. Thus $(r_j - \lceil n_j/b \rceil)b = s_j - g_j$. Combining with the fact $r_j \ne \lceil n_j/b \rceil, 0 \le g_j \le b-1$, and s_j is positive, we have $s_j - g_j$ is a positive multiple of b. From $n_j = (r_j - s_j)b + s_j(b-1)$, we can rewrite $n_j = (r_j - s_j + b - 1)b + (s_j - b)(b-1)$. Since $s_j - r$ is a positive multiple of b, we have $s_j - b$ is nonnegative. Thus we can partition X_j into $r_j - s_j + b - 1$ color classes of size b and $s_j - b$ color classes of size $b-1$. That is, we can partition X_j into $r_j - 1$ color classes of size b or $b-1$. Since we can partition other X_is into r_i color classes of size b or $b-1$ and $(\sum_{i\ne j} r_i) + (r_j - 1) = (\sum_{i=1}^{k} r_i) - 1 = r-1$, the graph G has an equitable $(r-1)$-coloring.

CASE 2: $r_i = \lceil n_i/b \rceil$ for each i. Since $r > p$ and the condition of d, we have $d > b$. Thus b violates conditions (i) and (ii) of d in Definition 1. Consequently, there are at least $k-1$ of n_is which are a multiple of b and $n_j/\lfloor n_j/b \rfloor \le b+1$ for each j. Without loss of generality, we assume $n_i = r_i b$ for each $i \ge 2$.

SUBCASE 2.1: $n_1 \ne r_1 b$. Then $b < n_1/\lfloor n_1/b \rfloor = n_1/(\lceil n_1/b \rceil - 1) = n_1/(r_1 - 1)$. Since b violates condition (ii), we have $n_1/(r_1-1) = n_1/\lfloor n_1/b \rfloor \le b+1$. Thus $b < n_1/(r_1-1) \le b+1$. Consequently, we can partition n_1 into $r_1 - 1$ color classes of size b or $b+1$. Combining with r_i color classes of X_i of size b for $i \ge 2$, we have an equitable $(r-1)$-coloring.

SUBCASE 2.2 $n_i = r_i b$ for each i. If there is j such that $n_j/(r_j-1) \le b+1$, then we have an equitable $(r-1)$-coloring as in subcase 2.1. Thus we assume further that $n_i/(r_i-1) > b+1$ for each i. We claim that $b+1 = d$ and $\lceil n_i/b \rceil = \lceil n_i/(b+1) \rceil = \lceil n_i/d \rceil$. If the claim holds, we have $r = \sum_{i=1}^{k} \lceil n/b \rceil = \sum_{i=1}^{k} \lceil n/d \rceil = p$ which contradicts to the fact that $r > p$. Thus this situation is impossible.

To prove the claim, suppose to the contrary that n_i is divisible by $b+1$ for some i. Since $n_i = r_i b$, we have $r_i = t_i(b+1)$ for some positive integer t_i. Thus $n_i/(r_i-1) = t_i(b+1)b/(t_i(b+1)-1) = b + b/(t_i(b+1)-1) \le b+1$ which contradicts to the fact that $n_i/(r_i-1) > b+1$. Thus n_i is not divisible by $b+1$ for each i. Consequently, $b+1 = d$ by condition (i). Since $n_i = r_i b$ and $n_i/(r_i-1) > b+1$ for each i, we have $r_i = n_i/b > n_i/(b+1) > r_i - 1$. This leads to $r_i = \lceil n_i/b \rceil = \lceil n_i/(b+1) \rceil$. Thus, we have the claim and this completes the proof. □

Lemma 2 *Assume that $G = K_{n_1,\dots,n_k}$ has an equitable q-coloring and $p = p(q : n_1, \dots, n_k)$. Then G has no equitable $(p-1)$-coloring.*

Proof. Suppose to the contrary that G has an equitable $(p-1)$-coloring. Then a partite set, say X_1 of size n_1, is partitioned into at most $\lceil n_1/d \rceil - 1$ color classes and a partite set X_j of size n_j is partitioned into at least $\lceil n_j/d \rceil$ color classes. Now we have at least color class containing vertices in X_1 with size at least $d+1$. By (i) and (ii) in Definition 1, we investigate 2 cases.

CASE 1: there is some X_j partitioned into at least $\lceil n_j/d \rceil + 1$ color classes or there is some n_j with $j \geq 2$ which is not divisible by d. But then we have at least one color class containing vertices in X_j with size at most $d-1$. This contradicts to the fact the sizes of two color classes differ at most one.

CASE 2: each X_j with $j \geq 2$ has exactly $\lceil n_j/d \rceil$ color classes and n_j is divisible by d. Then $\lceil n_j/d \rceil = d$ for $j \geq 2$. Thus n_1 has $n_1/\lfloor n_1/d \rfloor > d+1$ by the condition (ii) of d in Definition 1. But X_1 is partitioned into at least $\lceil n_1/d \rceil - 1 = \lfloor n_1/d \rfloor$ color classes. Thus we have at least one color class containing vertices in X_1 with size at least $d+2$. But each color class containing vertices in X_j where $j \geq 2$ has size d. Thus G has no equitable $(p-1)$-coloring. $\Box$

From Lemmas 1 and 2, we have the following theorem.

Theorem 3 *Assume that $G = K_{n_1,\dots,n_k}$ has an equitable q-coloring. Then $p(q : n_1, \dots, n_k)$ is the minimum p such that G has an equitable r-coloring for each r satisfying $p \leq r \leq q$.*

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